Supplementary Figure S1: Characterization of Device 1. a, Schematic of device 1, consisting of a clean nanotube grown over a predefined trench. Source and drain contacts to the nanotube are made by a 5/25 nm W/Pt bilayer (dark grey). Two gates embedded in the oxide (red) are used to induce charges in the nanotube. A backgate (blue) is kept grounded. b, A scanning electron microscope (SEM) image of the actual device, taken after all measurements. The nanotube axis lies at an angle of 48 degrees relative to the magnetic field orientation. c, $I_{\text{meas}}$ at a $V_{sd} = 1 \text{ mV}$ as a function of the two gate voltages. d, Magnetic field dependence of $I_{\text{meas}}$ along a diagonal line cut $V_{g1} = V_{g2} = V_g$ in c. The device exhibits a minimum gap at $B_{\text{Dirac}} = 2.2 \text{ T}$ (white arrows), corresponding to a crossing through the Dirac point of the graphene bandstructure by the $k_{\perp}$ quantization line. e, Plot of $I_{\text{meas}}$ (logscale) at $V_{sd} = 1 \text{ mV}$ showing the gate voltages corresponding to the first electrons in the quantum dot.
Supplementary Figure S2: Stability diagram showing Coulomb diamonds of Device 1. a, Differential conductance of device 1 showing the Coulomb diamonds of the first electrons, the empty device, and the threshold for hole conduction, taken at $B = 0$. For larger gate voltage, a four-fold pattern of Kondo resonances is observed, together with strong instabilities in the Coulomb diamonds which we attribute to mechanical excitation of mechanical resonances of the suspended nanotube by single-electron tunnelling [33]. Due to the lack of p-n junction barriers for holes, the hole doped region shows Fabry-Perot type oscillations. For the hole doped device, and for large electron numbers, we estimate the single-particle energy of the confined states to be $\sim 5$ meV. b, The same data as in a with the contrast enhanced in order to clearly show the Coulomb diamond of the first electron.
Supplementary Figure S3: Excited states of the 2e charge state in Device 1 Differential conductance vs. $V_G$ and $V_{SD}$ for the 1e to 2e transition, in which excited states could be resolved. The dashed lines indicates excited states we identify as the single-particle energy splitting $\Delta E_{SP}$. For the 1e-2e transition, we extract $\Delta E_{SP} = 11$ meV. From $\Delta E = \hbar v_F/(2L)$, we estimate the size of the quantum dot $L \sim 200$ nm. From the measured angle in the SEM image, the total length of the nanotube over the trench is $\sim 400$ nm. This implies a 100 nm length for the pn depletion region and p doped regions from the work function induced doping for the 1e quantum dot. For higher electron numbers, and similarly for holes, the single particle energy drops to $5$ meV (see figure S4), implying a confinement length responding to the full length of the suspended nanotube. The white arrow indicates the position of a faint excited state with an energy $\sim 3$ meV, consistent with the spin-orbit splitting we observe from the ground state measurements.
Supplementary Figure S4: Evidence for spin-orbit splitting in magnetic field spectroscopy of excited states in Device 1. A colourscale plot showing $dI/dV_g$ as a function of magnetic field for the 1e/2e transition of device 1 taken at $V_{SD} = 5.5$ mV. Excited states of the 2e ground state appear as positive peaks in $dI/dV_g$. The dashed line indicates the magnetic field dependence of a 2e excited state consistent with the expected spectrum from spin-orbit splitting. From the excited state data, we extract a spin-orbit splitting $\Delta_{SO} = 2.9$ meV, lower than that from the ground state measurements. This difference can arise from a difference between the excited state energies of the 2e state compared to the ground state energy of 3e from electron interactions. The origin of the extra excited state running parallel to the ground state for fields less than $B_1$ is not understood. The value of $\Delta_{SO}$ from the excited state measurement is, similar to that from the ground state measurements, an magnitude larger than the expected maximum $\Delta_{SO}^{max} = 106 \mu$eV expected from theory (see table S1). Excited states in the 0/1e transition did not show clear visibility, and those of higher states were masked by instabilities we attribute to mechanical excitation of the suspended nanotube (also visible here above 6T).
Supplementary Figure S5: Spin-orbit split states in Device 2. Magnetic field dependence of the first two electrons in device 2, showing the signature of the nanotube spin-orbit coupling with $\Delta_{SO} = 1.5$ meV.
Supplementary Figure S6: Spin-orbit spectrum of the first shell in Device 2. Extracted ground state energies of the first four electrons in device 2. Device 2 also shows a large spin orbit coupling with a dominant Zeeman-type contribution. Note that the flat behaviour of the ground states at zero magnetic field is a measurement artifact from the magnetic field controller, see text for discussion.
Supplementary Figure S7: Magnetic field dependence of the first four Coulomb peaks in Device 2. Coulomb blockade current vs. magnetic field and gate voltage for device 2, used to extract the ground state energies of the first shell, $V_{sd} = 1$ mV. The data in figure S6 is a zoom of the data here.
Supplementary Figure S8: Stability diagram of Device 2 in few-electron regime. a, Differential conductance of device 2 showing the Coulomb diamonds of the first electrons, the empty device, and the threshold for hole conduction, taken at $B = 0$. 
Supplementary Figure S9: Coulomb peak data for Shells 2 and 3 of Device 1. Measurements of $I_{\text{meas}}$ vs. $V_g$ and $B$ on device 1 that are used to extract the energy spectra shown in figure 3 of the main text, taken at $V_{sd} = 70 \mu$V.
Supplementary Figure S10: High resolution datasets in Device 1 of spin-orbit states. High resolution datasets of the 1e and 2e Coulomb peaks of device 1 with \( V_{SD} = 1 \) mV, showing behaviour at low magnetic fields. Here, the low field behaviour is not affected by artifacts in the first gate sweep first gate sweep at because of a slow sweep rate of the gate that provided sufficient time for the magnetic field controller to settle before reaching the position of the Coulomb peak.
Supplementary Figure S11: Characterization of Device 3.  

**a,** Schematic of device 3. The total device length is 600 nm. The device includes 5 local gates embedded in oxide under the suspended nanotube. In the measurements, the outermost gates ($V_{GL}$, $V_{GR}$) are used to tune the electron/hole number in a (p,n) type double quantum dot, while the inner three gates are used to tune the interdot tunnel barrier.  

**b,** An overview of gate space, indicating the (p,p), (p,n), (n,p), and (n,n) regions of gate space, and the identification of the (0,0) configuration.  

**c,** A color scale plot of the measured current as a function of $V_{GL}$ and $V_{GR}$ at $V_{SD} = 10$ mV. The boxes outlined by dashed lines show the triple points used to track the ground state energies in figure S12 and figure 4 of the main text.
Supplementary Figure S12: Energy spectra of the first electron shell and hole shell of Device 3. Extended datasets from figure 4 of the main text, showing the ground states of the first four electrons and the first four holes, extracted from the motion of the triple points indicated in the dashed boxes in figure S11 with magnetic field. a, Double-dot stability diagrams taken at $V_{sd} = 5$ mV. b, Expected spectra for the first four electrons (solid lines), as well as spectra from the next higher shell (dashed lines). c,d, Magnetic field dependence of gate space cuts (white dashed lines in a) for the first four electrons c and holes d. Similar to previous works [8], the magnetic field dependence of the third and fourth electrons/holes does not follow exactly the single-shell spin orbit spectrum (solid lines in b), but instead show extra crossings from downward moving levels in higher shells (dashed lines in b).
**Supplementary Table S1: Summary of previous spin-orbit measurements.**

<table>
<thead>
<tr>
<th>Reference</th>
<th>$\mu_{orb}$</th>
<th>$d^*$</th>
<th>$\Delta_{SO}^{\text{max}}$ theory</th>
<th>$\Delta_{SO}$ observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kuemmeth et al. [8]</td>
<td>1.55 meV/T</td>
<td>7.0 nm</td>
<td>110 $\mu$eV</td>
<td>370 $\mu$eV (1e)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>210 $\mu$eV (1h)</td>
</tr>
<tr>
<td>Jespersen et al. [8]</td>
<td>0.63 meV/T</td>
<td>2.9 nm</td>
<td>168 $\mu$eV</td>
<td>150 $\mu$eV (many electrons)</td>
</tr>
<tr>
<td>Jespersen et al. [31]</td>
<td>0.87 meV/T</td>
<td>5.3 nm</td>
<td>$\dagger$ 146 $\mu$eV</td>
<td>200 $\mu$eV (many electrons)</td>
</tr>
<tr>
<td>Churchill et al. [9]</td>
<td>0.33 meV/T</td>
<td>1.5 nm</td>
<td>520 $\mu$eV</td>
<td>170 $\mu$eV (1e)</td>
</tr>
<tr>
<td>Jhang et al. [10]</td>
<td>0.33 meV/T$^\diamond$</td>
<td>1.5 nm$^{**}$</td>
<td>520 $\mu$eV</td>
<td>2500 $\mu$eV$^{\dagger\dagger}$</td>
</tr>
<tr>
<td>Device 1</td>
<td>1.6 meV/T</td>
<td>7.2 nm</td>
<td>106 $\mu$eV</td>
<td>3400 $\mu$eV (1e)</td>
</tr>
<tr>
<td>Device 1</td>
<td>1.6 meV/T</td>
<td>3 nm$^\ddagger$</td>
<td>260 $\mu$eV</td>
<td>3400 $\mu$eV (1e)</td>
</tr>
<tr>
<td>Device 2</td>
<td>1.5 meV/T</td>
<td>6.8 nm</td>
<td>116 $\mu$eV</td>
<td>1500 $\mu$eV (1e)</td>
</tr>
<tr>
<td>Device 3</td>
<td>0.9 meV/T</td>
<td>4.1 nm</td>
<td>190 $\mu$eV</td>
<td>1700 $\mu$eV (1e)</td>
</tr>
<tr>
<td>Device 3</td>
<td>0.8 meV/T</td>
<td>3.7 nm</td>
<td>208 $\mu$eV</td>
<td>1300 $\mu$eV (1h)</td>
</tr>
</tbody>
</table>

* Estimated from the observed orbital magnetic moment, ignoring effects of $k_{||}$, unless otherwise noted.

† The value of the diameter for this entry is based on a detailed analysis of $\mu_{orb}$ as a function of shell number performed by the authors.

‡ The diameter for this entry is based on the observed AFM height of the nanotube.

$^\diamond$ Orbital moment implied from AFM height of the nanotube.

$^{**}$ Diameter from AFM.

$^{\dagger\dagger}$ Implied from bulk bandgap measurements.
Supplementary Note 1: Characterization of Device 1

A schematic of Device 1 is shown in figure S1a. Similar to previous studies[22], we make a clean suspended carbon nanotube quantum dot by growing the nanotube across a pre-defined structure in the last step of the fabrication. A SEM image of the actual device (taken after all measurements were completed) is shown in figure S1b. As we do not control the direction of the nanotube growth, it often crosses the trench at an angle, as can be seen in this device. From AFM measurements, we estimate the nanotube diameter to be 3 nm.

We apply a d.c. voltage across the source and drain of the device and measure the current through the nanotube as we sweep the gates, as shown in figure S1c. In the upper left corner of the plot, the gates dope the center of the nanotube with holes. Near the edge of the device, the gate electric fields are screened by the ohmic contact metal; here, the doping is set by the work function difference between the metal ($\Phi_{Pt} \sim 5.6$ eV) and the nanotube ($\Phi_{CNT} \sim 4.9$ eV), resulting in a gate-independent hole doping at the edge of the trench. This, combined with hole doping of the suspended segment from the gates, results in a $p'pp'$ configuration in the upper left corner of figure S1c. In this region, we observe only weak modulations of the conductance which does not vanish between peaks, indicating a highly transparent interface between the Pt metal and clean nanotube. In the lower right corner of figure S1c, the gates induce electrons in the suspended segment, giving a $p'n'p'$ doping profile. Electrons occupy a quantum dot with tunnel barriers defined by p-n junctions [22], in which we can count the number of carriers starting from zero, shown in figure S1e.

Figure S1d shows $I_{meas}$ vs. $V_g$ taken along the dotted line in S1c as a function of an external magnetic field applied in the plane of the sample, perpendicular to the trench. The distance in gate voltage between the onset of electron and hole current is a measure of the electronic bandgap of the nanotube. In carbon nanotubes, a magnetic field component parallel to the nanotube axis shifts the quantization condition of the states circling the circumference ($k_\perp$) by an Aharonov-Bohm flux, and therefore reduces the nanotube bandgap (see figure 2a of main text). For sufficiently large magnetic fields, the $k_\perp$ quantization line will cross the Dirac point of the graphene bandstructure and the bandgap begins to increase again. In our device, this occurs at a magnetic field of $B_{Dirac} = 2.2$ T, indicated by white arrows in figure S1d. This implies a contribution to the electronic bandgap $E_{gap}^{k_\perp} = 2\hbar v_F \Delta k_\perp \sim 7$ meV arising from the shift of the $k_\perp$ quantization line. In this sense, our nanotube is very close to the metallic condition in which the $k_\perp$ quantization line passes directly through the center of the Dirac cone. This is a very different regime compared to previous devices where the nanotube spin orbit coupling was studied [8,11], in which no such evidence of a low Dirac field was seen. Similar to previous studies where low Dirac fields were reported[28], the bandgap do not vanish at the Dirac point. We observe a residual gap in the transport data at the Dirac point of about 80 meV, measured by subtracting the average of the addition energies from the first electron and the first hole from the addition energy of the empty quantum dot.
Supplementary Note 2: Spin orbit splitting in Device 2

In figures S6-S9, we present the magnetic field dependence of the ground states of the first four electrons in a second nearly metallic carbon nanotube (device 2). Device 2 is similar in design to device 1, but includes only a backgate. The trench length is 800 nm. In device 2, we observe a Dirac field of 0.8 T, an orbital magnetic moment $\mu_{\text{orb}} = 1.5$ meV/T, and a spin orbit splitting $\Delta_{SO} = 1.5$ meV.
Supplementary Note 3: Model for a nearly metallic nanotube with spin-orbit coupling

In order to calculate the spectra plotted in figures 2g and h of the main text, we use a model of the nanotube based on the graphene bandstructure with a parallel magnetic field. In a basis of spin and valley eigenstates in which the spin direction is defined parallel to the axis of the nanotube, the Hamiltonian consists of a 4x4 matrix with only diagonal elements given by:

\[ E(v, s, B) = \sqrt{(E_{k\parallel} + vs\Delta_{SO} + v\mu_{orb}B)^2 + E_{k\perp}^2 + vs\Delta_{Zeeman} + \frac{1}{2}sg\mu_B B} \]  

(S1)

Here, \( v \) and \( s \) take on values ±1 depending on the electron spin and the valley it occupies, \( E_{k(\parallel,\perp)} = \hbar v_F k_{(\parallel,\perp)} \) where \( k_{(\parallel,\perp)} \) are the momentum of the electron relative to the Dirac points in the directions parallel and perpendicular to the axis of the nanotube, and \( \Delta_{SO} \) and \( \Delta_{Zeeman} \) are the orbital and Zeeman type spin orbit splittings at \( k_{||} = 0 \) (\( \alpha \) and \( \beta \)). These diagonal elements correspond to the energies plotted in figure 3. In the calculations, we have chosen to make the total spin orbit coupling either purely orbital or purely Zeeman for illustrative purposes, and have used the following parameters: \( \Delta_{SO} = 2 \text{ meV}, E_{k\parallel} = 1 \text{ meV}, E_{k\perp} = 2 \text{ meV}, \) and \( \mu_{orb} = 0.9 \text{ meV/T}. \)

Including the observed 48 degree misalignment of the magnetic field to the nanotube axis, the Zeeman splitting Hamiltonian \( g\mu_B \vec{B} \cdot \vec{S} \) is no longer diagonal in this basis, and the eigenstates are mixtures of the four basis states described above. However, because the Bohr magneton is small compared to the orbital magnetic moment, this effect is weak and does not result in qualitative different spectra.

The Zeeman-type contribution to the spin-orbit splitting, according to current theoretical estimates, is expected to be larger than the orbital-type contribution by as much as a factor of 4, except for in nanotube chiralities where it vanishes or is small due to the \( \cos(3\theta) \) term (\( \theta = 0 \) corresponding to a zigzag nanotube). It is an open question, however, why the spin-orbit splitting we observe in devices 1 and 2 is so dominantly of the Zeeman-type, with little indication of an orbital contribution.
Supplementary Note 4: Discussion of summary table of previous spin-orbit splitting measurements

In Supplementary Table S1, we summarize in a table our measurements together with other measurements of the spin-orbit coupling reported in literature. As described in the main text, we use the formula for the nanotube spin-orbit splitting from [27], given by:

$$H_{SO}^{cv} = \alpha S^z \sigma_1 + \tau \beta S^z$$  \hspace{1cm} (S2)

with the orbital contribution given by:

$$\Delta_{SO}^{orb} = \alpha = -0.08 \text{ meV } \text{nm}$$ \hspace{1cm} (S3)

and the Zeeman contribution given by:

$$\Delta_{SO}^{Zeeman} = \beta = -0.31 \text{ meV } \text{nm}$$ \hspace{1cm} (S4)

where \(r\) is the radius of the nanotube. For the maximum theoretical value, we choose \(\theta = 0\), giving:

$$\Delta_{SO}^{max} = \frac{780 \mu\text{eV}}{\text{d (in nm)}}$$ \hspace{1cm} (S5)

In order to provide a consistent comparison, we have estimated the (minimum) diameter using the observed value of the orbital magnetic moment \(\mu_{orb}\). Assuming a Fermi velocity of \(0.9 \times 10^6\) m/s, \(\mu_{orb}\) is given by:

$$\mu_{orb} = \frac{d v_F}{4} = 220 \mu\text{eV} / \text{eV} \times \text{d (in nm)}$$ \hspace{1cm} (S6)

where \(v_F\) is the Fermi velocity of the graphene bandstructure, which we take here as \(0.9 \times 10^6\) m/s. Here, we assume \(v_{F\perp} = v_F\), and therefore have not accounted for the reduction of \(\mu_{orb}\) from a finite \(k_{||}\) [31]. The resulting estimates of \(d\) from \(\mu_{orb}\) represent a lower bound on the diameter (and thus also an upper bound on \(\Delta_{SO}^{max}\)).

We have also included three entries in which we calculate \(\Delta_{SO}^{max}\) based on a different estimate of the diameter. These three entries correspond to the diameter \(d = 3\) nm we estimate from AFM measurements on device 1, the diameter \(d = 5.3\) nm estimated by [31] from an extensive analysis of \(\mu_{orb}\) as a function of gate voltage, and the diameter \(d = 1.5\) nm measured by Jhang et al. [10]. Note that the tapping-mode AFM measurement of the diameter may underestimate the diameter of single-wall carbon nanotubes due to compression forces from the AFM tip.

The measurements referred to in the table were performed by tracking the electronic states of indi-
individual levels in a quantum dot at low temperatures, except for the measurements of Jhang et al. [10]. The values in [10] are based on measurements of the nanotube bandgap implied from device conductance near the bandgap as a function of magnetic field at different fixed gate voltages, together with the nanotube diameter as measured by AFM. The devices measured here and those measured by Kuemmeth et al. [8] were made using clean nanotubes grown in the last step of the fabrication, while the other measurements were performed on nanotubes which were grown first and subsequently underwent processing in the cleanroom.

Finally, we also note that when using $\mu_{orb}$ to estimate the nanotube diameter, we obtain a number that is not only larger than the AFM measurement for device 1, but also larger than the largest diameter expected for single wall carbon nanotubes in, for example, transmission electron microscope studies. This is also the case for many of the devices in Table 1. Such a discrepancy was also noted by earlier authors [31], and remains unresolved. One suggestion of the authors of [31] was a renormalization of the Fermi velocity. Such a renormalization could arise from, for example, discrepancies between the experimental tight binding parameters of carbon nanotubes and those obtained from \textit{ab initio} calculations.
Supplementary Note 5: Artifacts in extracted ground state energies at $B < 0.15$ T

Note that there is glitch in the first line of the data set in figure S6. This artifact is also present to a lesser degree figures 1(c)-(f) and the resulting extracted energies in figures 3(a)-(c) of the main text. This glitch results in an artifact in the resulting extracted ground state energies plotted in figure S5 in the form of a flat slope for $B < 150$ mT. The glitch and resulting artifacts arise from the inability of our magnetic controller to track the setpoint field during faster magnetic field sweeps. The effects of these artifacts are limited to the first gates sweep (row) of the Coulomb peak magnetic field dependence data. These artifacts have been accounted for in the estimation of the error bar on $\Delta_{SO}$.

In order to demonstrate that these artifacts are not obscuring possible other phenomena at very low magnetic fields, we have also included high resolution datasets in figure S10 for the data in figure 1(c) and 1(d) of the main text. Here, the gate was swept sufficiently slowly that the magnet controller had time to settle before the gate voltage reached the position of the first Coulomb peak, and thus the artifacts are not present.
Supplementary Note 6: Device 3 characterization and analysis

In this section, we present a basic characterization of device 3 (figure S11), together with measurements the magnetic field dependence of the ground state energies of the first four electrons and first four holes in the device (figure S12), and discuss the extraction of the ground state energies from the magnetic field dependence of the gate-space cuts through the triple-point triangles.

By tracking the gate voltage position of any fixed point on the triple-point bias triangles as a function of magnetic field, we can independently track the ground state energy of the left and right dot in the double quantum dot device. This is analogous to the tracking of the ground states of a single quantum dot by following the Coulomb peak position with magnetic field. To make this concrete, we illustrate this in the context of upper left bias triangle in figure 4a of the main text, corresponding to the (3h,1e) ↔ (2h,0e) transition. In the case that there is very small crosstalk capacitance from the left gate to the right dot (as is the case in figure 4a of the main text where the edges of the triple-point bias triangle are nearly vertical), vertical shifts of the bias triangle arise from shifts in the 3h ground state, while shifts in the 1e ground state shift the bias triangle horizontally. In measuring the shift of the bias triangle, it is equivalent to track any fixed point on the triangle. We choose to extract the ground state energies by following a point near the tip of the triangle, as the current on the baseline in our device is weak due to weakly tunnel-coupled ground states.
Supplementary References